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Constants of Motion for Several One-Dimensional Systems and Problems Associated with Getting their Hamiltonians

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The constants of motion of the following systems are deduced: a relativistic particle with linear dissipation; a no-relativistic particle with a time explicitly depending force; a no-relativistic particle with a constant force and time depending mass; and a relativistic particle under a conservative force with position depending mass. The Hamiltonian for these systems, which is determined by getting the velocity as a function of position and generalized linear momentum, can be found explicitly at first approximation for the first system. The Hamiltonians for the other systems are kept implicitly in their expressions for their constants of motion.

KEY WORDS: constant of motion.
PCAC: 03.20. + i 03.65. Ca

1. INTRODUCTION

The constant of motion of a dynamical system, which has an equivalent interpretation of the energy of the system, has received attention lately for three reasons. First, there is an interest in studying dissipative systems (Okubo, 1981; Cantrijn, 1982). Second, there are some well-known problems with the Hamiltonian formalism (Yang, 1981). Finally, there is the possibility of making a quantum mechanics formulation based on the constant of motion concept (López, 1998, 2002). The constant of motion concept, besides its obvious usefulness in classical mechanics (Goldstein, 1950), can have great deal of importance in quantum mechanics and statistical physics for system without well-defined Hamiltonian (López, 1999a). In particular, when one studies relativistic systems with no-conservative forces or

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systems with time depending mass or systems with position depending mass, the concept of constant of motion appears more naturally than the concept of Hamiltonian. In this paper we analyze four dynamical systems and find their constant of motion. Even though, this problem seems to be simple to analyze, we want to point out the importance of the constant of motion and the difficulty to get the Hamiltonian for several systems. These constants of motion are selected such that when some interaction is neglected, they are reduced to the usual concept of energy. The Hamiltonian associated to the system is deduced whenever is possible to do that. The paper is organized as follows: first we study a relativistic system with linear dissipation and with a constant external force. For this system, the constant of motion is given in general, and the Hamiltonian is obtained for weak dissipation. Then, we study a no-relativistic system with an external time explicitly depending force, where only the constant of motion is given. In the same way, we find a constant of motion for a no-relativistic system with a constant force and with a time depending mass. Similarly and finally, we obtained the constant of motion of a relativistic system with position depending mass and a force proportional to this mass.

2. CONSTANT OF MOTION OF A RELATIVISTIC PARTICLE WITH LINEAR DISSIPATION

The motion of a relativistic particle with rest mass "m" affected by a constant force "f" and a linear dissipation force is described by the equation

$$\frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) = f - \alpha v, \tag{1}$$

where v is the velocity of the particle, c is the speed of light, and α is the parameter which characterizes the dissipation. Equation (1) can be written as the following autonomous dynamical system

$$\frac{dx}{dt} = v; \qquad \frac{dv}{dt} = \frac{f}{m}(1-\beta v)(1-v^2/c^2)^{3/2}, \tag{2}$$

where β has been defined as $\beta = \alpha/f$, and x is the position of the particle. A constant of motion of this system is a function $K_\beta(x, v)$ which satisfies the equation (López, 1999b)

$$v\frac{\partial K_\beta}{\partial x} + \frac{f}{m}(1-\beta v)(1-v^2/c^2)^{3/2}\frac{\partial K_\beta}{\partial v} = 0. \tag{3}$$

The solution of Equation (3), such that for β equal to zero one gets the usual expression for the relativistic energy,

$$\lim_{\beta\to 0} K_\beta = \frac{mc^2}{\sqrt{1-v^2/c^2}} - fx - mc^2, \tag{4}$$

is given by

$$K_\beta = -fx - mc^2 + \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \begin{cases} \frac{1+\beta v}{1-\beta^2c^2} + \frac{\beta c\sqrt{1-v^2/c^2}}{(\beta^2c^2-1)^{3/2}} \ln A_\beta(v) & \text{if } \beta > 1/c \\ \frac{1+\beta v}{1-\beta^2c^2} + \frac{\beta c\sqrt{1-v^2/c^2}}{(1-\beta^2c^2)^{3/2}} \arctan B_\beta(v) & \text{if } \beta < 1/c \\ \frac{1}{3}\left[\frac{v}{c} - \frac{1}{1-v/c}\right] & \text{if } \beta = 1/c \end{cases} \tag{5a}$$

where the functions $A_\beta(v)$ and $B_\beta(v)$ are defined as

$$A_\beta(v)\frac{2(\beta^2c^2-\beta v) + 2\beta c\sqrt{\beta^2c^2-1}\sqrt{1-v^2/c^2}}{1-\beta v} \tag{5b}$$

and

$$B_\beta(v) = \frac{\beta^2c^2-\beta v}{\beta c\sqrt{1-\beta^2c^2}\sqrt{1-\frac{v^2}{c^2}}} \tag{5c}$$

At first order on the dissipation parameter, this constant of motion can be written as

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - fx - mc^2 + \beta mc^3\left[\frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} - \arctan\left(\frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}}\right)\right]. \tag{6}$$

Now, using the known expression relating the constant of motion and the Lagrangian (Kobussen, 1979; Leubner, 1981; López,1996),

$$L = v\int \frac{K(x,v)\,dv}{v^2}, \tag{7}$$

this Lagrangian is calculated, bringing about the expression

$$L_\beta = -fx - mc^2 + \begin{cases} \frac{mc^2\sqrt{1-\frac{v^2}{c^2}}}{\beta^2c^2-1} + \frac{\beta c^2 vm}{\beta^2c^2-1}\ln\left[\frac{2(1+\sqrt{1-v^2/c^2})}{\beta v}\right] + \frac{mc^2G_\beta(v)}{4(\beta^2c^2-1)} & \text{if } \beta > 1/c \\ \frac{mc^2\sqrt{1-\frac{v^2}{c^2}}}{\beta^2c^2-1} + \frac{\beta c^2 vm}{\beta^2c^2-1}\ln\left[\frac{2(1+\sqrt{1-v^2/c^2})}{\beta v}\right] + \frac{mc^3\beta Q_\beta(v)}{(1-\beta^2c^2)^2} & \text{if } \beta < 1/c \\ \frac{mc^2}{3\sqrt{1-v^2/c^2}}\left[1 - \frac{v}{c} - \frac{2v^2}{c^2} + \frac{v}{c}\sqrt{1-\frac{v^2}{c^2}}R_\beta(v)\right] - \frac{mcvR_\beta(v)}{3} & \text{if } \beta = 1/c \end{cases} \tag{8}$$

where the functions G_β, Q_β and R_β are given in the appendix. The generalized linear momentum, $p = \partial L/\partial v$, has the following expression

$$p_\beta = \begin{cases} \frac{-mv}{(\beta^2c^2-1)\sqrt{1-\frac{v^2}{c^2}}} + \frac{\beta c^2 m}{\beta^2c^2-1}\ln\left[\frac{2\left(1+\sqrt{1-\frac{v^2}{c^2}}\right)}{\beta v}\right] - \frac{\beta mc^2}{(\beta^2c^2-1)\sqrt{1-\frac{v^2}{c^2}}} + A_\beta^{(1)} & \text{if } \beta > 1/c \\ \frac{-mv}{(\beta^2c^2-1)\sqrt{1-\frac{v^2}{c^2}}} + \frac{\beta c^2 m}{\beta^2c^2-1}\ln\left[\frac{2\left(1+\sqrt{1-\frac{v^2}{c^2}}\right)}{\beta v}\right] - \frac{\beta mc^2}{(\beta^2c^2-1)\sqrt{1-\frac{v^2}{c^2}}} + A_\beta^{(2)} & \text{if } \beta < 1/c \\ \frac{mc}{3(1-\frac{v}{c})\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{2v^2}{c^2} - \frac{2v}{c} - 1\right) & \text{if } \beta = 1/c \end{cases} \tag{9}$$

where the functions $A_\beta^{(1)}$ and $A_\beta^{(2)}$ are given in the appendix. As one can see from (9), it is not possible to express v explicitly as a function of p_β. Therefore, it is not possible to know explicitly the Hamiltonian of the system. However, for weak dissipation one can use (6) in (7) to get

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + fx + mc^2 + \beta mc^3 \arctan\left(\frac{v/c}{\sqrt{1-v^2/c^2}}\right). \tag{10}$$

The generalized linear momentum has the following expression

$$p = \frac{mv + \beta mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{11}$$

and from this expression, one gets

$$v = \frac{-\beta m^2c^2 + p\sqrt{p^2/c^2 + m^2 - \beta^2m^2c^2}}{p^2/c^2 + m^2}. \tag{12}$$

So, the Hamiltonian for weak dissipation can be written as

$$H = \frac{mc^2(p^2/c^2 + m^2)}{g_\beta(p)} - fx - mc^2 + \beta mc^3\Delta_\beta(p), \tag{13a}$$

where $g_\beta(p)$ and $\Delta_\beta(p)$ are defined as

$$g_\beta = \sqrt{\left(\frac{p^2}{c^2} + m^2\right)^2 - \beta^2m^2c^2 + 2\beta m^2 p\sqrt{\frac{p^2}{c^2} + m^2 - \beta^2m^2c^2} - \frac{p^2}{c^2}\left(\frac{p^2}{c^2} + m^2 - \beta^2m^2c^2\right)} \tag{13b}$$

and

$$\Delta_\beta = \frac{-\beta m^2c^2 + p\sqrt{\frac{p^2}{c^2} + m^2 - \beta^2 m^2 c^2}}{c g_\beta(p)} - \arctan\left[\frac{-\beta m^2c^2 + p\sqrt{\frac{p^2}{c^2} + m^2 - \beta^2 m^2 c^2}}{c g_\beta(p)}\right] \tag{13c}$$

Note that the function g_β has the following limit $\lim_{\beta\to 0} g_\beta(p) = m\sqrt{p^2/c^2 + m^2}$. Thus, (13a) has the usual Hamiltonian expression for the non dissipative case as β goes to zero.

3. CONSTANT OF MOTION FOR A TIME DEPENDING FORCE

The motion of a no-relativistic particle of mass "m" affected by a time depending force, $f(t)$, can be written as the following non-autonomous dynamical system

$$\frac{dx}{dt} = v; \qquad \frac{dv}{dt} = f(t)/m. \tag{14}$$

A constant of motion for this system is a function $K(x, v, t)$ such that it satisfies the following equation

$$v\frac{\partial K}{\partial x} + \frac{f(t)}{m}\frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0. \tag{15}$$

Solving this equation by the characteristics method (John, 1974), one gets the general solution given by

$$K(x, v, t) = G(C_1, C_2), \tag{16}$$

where G is an arbitrary function of the characteristics C_1 and C_2 which have the following expressions

$$C_1 = v - \frac{1}{m}\int f(t)\,dt, \tag{17a}$$

and

$$C_2 = x - vt + \frac{t}{m}\int f(t)\,dt - \frac{1}{m}\int\left(\int^t f(s)\,ds\right)dt. \tag{17b}$$

Let us choose $f(t)$ of the form

$$f(t) = f_o[1 + \epsilon g(t)], \tag{18}$$

where $g(t)$ is an arbitrary nonsingular function, and ϵ and f_o are parameters. Note that $\lim_{\epsilon\to 0} f(t) = f_o$, and in this limit the usual constant of motion is the energy,

$K_o = \lim_{\epsilon\to 0} K = mv^2/2 - f_o x$. In order to get this energy expression from our characteristics, one needs in (16) the following functionality $\lim_{\epsilon\to 0} G(C_1, C_2) = (mC_1^2/2 - f_o C_2)_{\epsilon=0}$. So, one can choose this functionality in (16) for $\epsilon \neq 0$, having the constant of motion given by

$$K = \frac{m}{2}[v - h_1(t)]^2 - f_o[x - vt + th_1(t) - h_2(t)], \tag{19}$$

where h_1 and h_2 have been defined as

$$h_1(t) = \frac{1}{m}\int f(t)\,dt, \tag{20a}$$

and

$$h_2(t) = \int h_1(t)\,dt. \tag{20b}$$

The expression (19) can also be written as

$$K = K_o(x, v) + V_\epsilon(v, t), \tag{21a}$$

where K_o and V_ϵ have been defined as

$$K_o(x, v) = \frac{1}{2}mv^2 - f_o x, \tag{21b}$$

and

$$V_\epsilon = -mvh_1(t) + f_o vt + \frac{1}{2}mh_1^2(t) - f_o th_1(t) + f_o h_2(t). \tag{21c}$$

It is not difficult to see that the following limit is satisfied

$$\lim_{\epsilon\to 0} V_\epsilon(v, t) = 0. \tag{21d}$$

In particular, for a periodic function,

$$g(t) = \sin(\Omega t), \tag{22}$$

one gets

$$K = K_o + \frac{\epsilon f_o v}{\Omega}\cos(\Omega t) + \frac{f_o^2\epsilon^2}{2m\Omega^2}\cos^2(\Omega t) - \frac{\epsilon f_o^2}{m\omega^2}\sin(\Omega t). \tag{23}$$

Since the expression (14) represents a non autonomous system, the possible associated Hamiltonian can not be a constant of motion, and the expression (7) can not be used to calculate the Lagrangian of the system, therefore, its Hamiltonian (López and Hernández, 1989). Naively, one can consider (14) as a Hamiltonian system and $H = p^2/2m - f(t)x/m$ as its associated Hamiltonian ($p = mv$), and $L = mv^2/2 + f(t)x/m$ as its associated Lagrangian. However, this procedure is difficult to justify and is not free from ambiguities.

4. CONSTANT OF MOTION OF A TIME DEPENDING MASS SYSTEM

The motion of a particle with a time depending mass and affected by a constant force can be described by the following no-autonomous dynamical system

$$\frac{dx}{dt} = v; \qquad \frac{dv}{dt} = \frac{f}{m} - \frac{\dot{m}}{m}v, \tag{24}$$

where f represents the constant force, $m = m(t)$ is the mass of the system, and $\dot{m}$ is its time differentiation (this type of systems appear for example in rockets or missiles motion studies). A constant of motion for this system is a function $K(x, v, t)$ which satisfies the equation

$$v\frac{\partial K}{\partial x} + \left[\frac{f}{m} - \frac{\dot{m}}{m}v\right]\frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0. \tag{25}$$

Solving (25) by the characteristics method, the general solution is gotten as

$$K(x, v, t) = G(C_1, C_2), \tag{26}$$

where G is an arbitrary function of the characteristics C_1 and C_2 which are defined as

$$C_1 = mv - ft \tag{27a}$$

and

$$C_2 = x - mv\int\frac{dt}{m(t)} + f\left[t\int\frac{dt}{m(t)} - \int\frac{t\,dt}{m(t)}\right]. \tag{27b}$$

If one assumes that the mass is constant, $m(t) = m_o$, the characteristic curves would be given by $C_1 = mv - ft$ and $C_2 = x - vt - ft^2/2m$. So, the functionality G which brings about the usual constant of motion (energy) would be given by $G = C_1^2/2m_o - fC_2 = mv^2/2 - fx$. Therefore, for the case where the mass depends explicitly on time,

$$m(t) = m_o g_\epsilon(t), \tag{28}$$

such that $\lim_{\epsilon\to 0} g_\epsilon = 1$, one chooses the following functionality

$$G(C_1, C_2) = \frac{1}{2m_o}C_1^2 - fC_2 \tag{29}$$

which brings about the constant of motion of the form

$$K_\epsilon(x, v, t) = K_{o\epsilon}(x, v, t) + W_\epsilon(v, t), \tag{30a}$$

where $K_{o\epsilon}$ and W_ϵ are given by

$$K_{o\epsilon} = \frac{m_o g_\epsilon^2}{2}v^2 - fx \tag{30b}$$

and

$$W_\epsilon = -g_\epsilon(t) f v t + \frac{f^2 t^2}{2m_o} + g_\epsilon(t) f v \Lambda_1(t) - \frac{f^2}{m_o}\Lambda_2(t). \tag{30c}$$

The functions $\Lambda_1(t)$ and $\Lambda_2(t)$ have been defined as

$$\Lambda_1(t) = \int \frac{dt}{g_\epsilon(t)} \tag{30d}$$

and

$$\Lambda_2(t) = t\Lambda_1(t) - \int \frac{t\,dt}{g_\epsilon(t)}. \tag{30e}$$

The functions $K_{o\epsilon}$ and W_ϵ have the following limits

$$\lim_{\epsilon\to 0} K_{o\epsilon} = \frac{1}{2}m_o v^2 - f x \tag{31a}$$

and

$$\lim_{\epsilon\to 0} W_\epsilon = 0. \tag{31b}$$

About these results, one has the same observation made on previous systems, that is, the constant of motion is a well-defined concept, in contrast, the Lagrangian and the Hamiltonian are no free from ambiguities.

5. CONSTANT OF MOTION OF A POSITION DEPENDING MASS SYSTEM

The motion of a relativistic particle of position depending mass, $m(x)$, and affected by a conservative force $f(x)$ is given by the equation

$$\frac{d}{dt}\left(\frac{m(x)v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = f(x), \tag{32}$$

where v is the velocity of the particle. This equation can be written as the following autonomous system

$$\frac{dx}{dt} = v \tag{33a}$$

and

$$\frac{dv}{dt} = \frac{f(x)}{m}\left(1-\frac{v^2}{c^2}\right)^{3/2} - \left(1-\frac{v^2}{c^2}\right)\frac{v^2 m_x}{m}, \tag{33b}$$

where m_x is the differentiation of the mass m with respect to the position (this type of systems may be important for mass particle oscillations along its trajectory, like neutrino mass oscillation problem). A constant of motion for this system is a function $K(x, v)$ satisfying the equation

$$v\frac{\partial K}{\partial x} + \left[\left(1 - \frac{v^2}{c^2}\right)^{3/2} \frac{f(x)}{m} - \left(1 - \frac{v^2}{c^2}\right) \frac{v^2 m_x}{m}\right] \frac{\partial K}{\partial v} = 0. \tag{34}$$

The general solution of (34) is given by

$$K(x, v) = G(C), \tag{35}$$

where C is the characteristic curve obtained from the solution of the following equation

$$\frac{dx}{v} = \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2} \frac{f(x)}{m} - \left(1 - \frac{v^2}{c^2}\right) \frac{v^2 m_x}{m}}. \tag{36}$$

From this expression, one can clearly see that this equation can be integrated for special cases only. For example, assuming $f(x)$ of the form

$$f(x) = -\alpha m_x c^2, \tag{37}$$

where α is a constant. Using (37) in (36) and the variable $\xi = \sqrt{1 - v^2/c^2}$, the integration can be done, getting the following characteristic curve (in terms of the variable v)

$$C_\alpha = m\sqrt{\frac{v^2/c^2 + \alpha\sqrt{1 - v^2/c^2}}{1 - v^2/c^2}} \left(\frac{\sqrt{\alpha^2 + 4} - \alpha + 2\sqrt{1 - v^2/c^2}}{\sqrt{\alpha^2 + 4} + \alpha - 2\sqrt{1 - v^2/c^2}}\right)^{\frac{\alpha}{2\sqrt{\alpha^2+4}}}. \tag{38}$$

Note, from (7), that $\alpha = 0$ represents the case of a relativistic free particle with position depending mass, and from (38) one gets the following limit

$$\lim_{\alpha \to 0} C_\alpha = \frac{m(x)\, v}{c\sqrt{1 - v^2/c^2}}. \tag{39}$$

Thus, one can choose G as $G(C_\alpha) = c^2 C_\alpha^2 / 2m_o$, where m_o is the value of m at $x = 0$. So, the constant of motion is given by

$$K_\alpha = \left(\frac{m^2(x)}{2m_o}\right) \frac{v^2 + \alpha c^2\sqrt{1 - v^2/c^2}}{1 - v^2/c^2} \left(\frac{\sqrt{\alpha^2 + 4} - \alpha + 2\sqrt{1 - v^2/c^2}}{\sqrt{\alpha^2 + 4} + \alpha - 2\sqrt{1 - v^2/c^2}}\right)^{\frac{\alpha}{\sqrt{\alpha^2+4}}}. \tag{40}$$

In addition, if $m(x)$ is chosen of the form

$$m(x) = m_o g_\epsilon(x), \tag{41}$$

where $\lim_{\epsilon\to 0} g_\epsilon(x) = 1$, one would have the following expected limit

$$\lim_{\substack{\alpha\to 0\\ c\to\infty\\ \varepsilon\to 0}} K_\alpha = \frac{1}{2} m_o v^2. \tag{42}$$

For example, let $m(x)$ be of the form

$$m(x) = m_o(1 + \epsilon \sin(kx)). \tag{43}$$

Then, the constant of motion is given by

$$K_\alpha(x, v) = \frac{m_o c^2}{2} (1 + \epsilon \sin(kx))^2 F_\alpha\left(\frac{v}{c}\right), \tag{44}$$

where the function F_α is defined as

$$F_\alpha\left(\frac{v}{c}\right) = \frac{v^2/c^2 + \alpha\sqrt{1 - v^2/c^2}}{1 - v^2/c^2} \left(\frac{\sqrt{\alpha^2+4} - \alpha + 2\sqrt{1 - v^2/c^2}}{\sqrt{\alpha^2+4} + \alpha - 2\sqrt{1 - v^2/c^2}}\right)^{\frac{\alpha}{\sqrt{\alpha^2+4}}}. \tag{45}$$

Given the initial condition (x_o, v_o), this constant is determined, and the trajectory in the space (x, v) can be traced. On the other hand, for the system (33) and for the particular case seen above, the expression (7) can be used, in principle, to obtain the Lagrangian of the system. However, the integration can not be done in general. Even more, if this Lagrangian is explicitly known and the generalized linear momentum is calculated, one can not know $v = v(x, p)$, in general. Thus, the Hamiltonian of this system can not be known explicitly.

6. CONCLUSIONS

We have given constants of motion for several one-dimensional systems. These constants of motion were chosen such that they can have the usual energy expression when the parameter which characterizes the no-conservative interaction goes to zero. For a relativistic particle with linear dissipation, its constant of motion was deduced in general, but its Hamiltonian was explicitly given only for weak dissipation. The problem to get the Hamiltonian was due, in general, to the fact that it is not possible to obtain the velocity explicitly as a function of the linear momentum and position from the expression $p = p(x, v)$. For a no-relativistic time depending system, for a no-relativistic time depending mass system affected by a constant force and for a mass position depending system affected by a constant force, only the constants of motion were given. The problem of getting their Hamiltonians is essentially the same as the first system.

APPENDIX

The function $G_\beta(v)$ is given by

$$G_\beta(v) = -2\sqrt{2+2\beta c}\; v\; \arctan\left(\frac{v\sqrt{1+\beta c}}{c\sqrt{2}}\right) + 4\beta cv\;\ln\left(\frac{v/c}{\beta v - 1}\right)$$
$$-4c\ln\left(\frac{2\beta \mathrm{c}(-1+\beta c+\sqrt{\beta^2c^2-1}\sqrt{1-v^2/c^2}}{1-\beta v}\right)$$
$$-\sqrt{2+2\beta c}\, v H_\beta^+(v) + \sqrt{2+2\beta c}\, v h_\beta^-(v), \tag{A1}$$

where the function with h_β^s with $s=\pm1$ is given by ($\gamma^{-1}=\sqrt{1-v^2/c^2}$)

$$h_\beta^s(v) = \ln\left[\frac{4v\sqrt{(\beta c)^2-1}+s2\beta c^2\sqrt{2+2\beta c}\gamma^{-1}+2c\sqrt{2+2\beta c}(s\sqrt{\beta c)^2-1}-s\gamma^{-1})}{(\beta c-1)\sqrt{\beta^2c^2-1}(sc\sqrt{2+2\beta c}+v+\beta cv)}\right]. \tag{A2}$$

The function $Q_\beta(v)$ is given by

$$Q_\beta(v) = \sqrt{1-(\beta c)^2}\arctan\left(\frac{\beta c - v/c}{\sqrt{1-(\beta c)^2}\gamma^{-1}}\right)$$
$$+\beta cv\sqrt{1-(\beta c)^2}\;\ln\left(\frac{2\left(\frac{\beta c-v/c}{1-(\beta c)^2}+\frac{1}{\gamma\sqrt{1-(\beta c)^2}}\right)}{1-\beta v}\right)$$
$$+(1-\beta^2c^2)\;\ln\left(\frac{2c(1-\beta^2c^2+(1-\beta^2c^2)\sqrt{1-v^2/c^2}}{v(1-\beta^2c^2)^{3/2}}\right). \tag{A3}$$

The function $R_\beta(v)$ is given by

$$R_\beta(v) = \;\ln\left(\frac{2c(1+\gamma^{-1})}{v}\right). \tag{A4}$$

The function $A_\beta^{(1)}$ is given by

$$A_\beta^{(1)}(v) = \frac{mc^3v}{\beta^2c^2-1}\left\{\frac{\beta}{v} - \frac{\beta^2}{\beta v-1} - \frac{1+\beta c}{2c^2\left(1-\frac{(1+\beta c)v^2}{2c^2}\right)}\right.$$
$$+\frac{(\beta^2c^2-1)f_1(v)}{g_1(v)(-\sqrt{2}\;c\sqrt{1+\beta c}+v+\beta cv)^2}$$

$$
-\frac{(\beta^2c^2-1)f_2(v)}{g_2(v)(\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)^2}-\frac{f_3(v)}{g_3(v)(1-\beta v)^2}
+\frac{1}{v^2}\ln\left(\frac{2(-\beta c+\beta^2c^2)+2\beta c\sqrt{(\beta c)^2-1}\,\gamma^{-1}}{1-\beta v}\right)\Bigg\} \tag{B1}
$$

where f_1, f_2, f_3, g_1, g_2 and g_3 are defined as

$$
\begin{aligned}
f_1(v) = {} & \frac{2(1+\beta c)(\sqrt{2}\,c\sqrt{1+\beta c}-2v)}{2(\beta c-1)}+\frac{4}{\beta c-1}(-\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)\\
& +\frac{2\sqrt{2}\sqrt{1+\beta c}\;v(-\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)}{c\sqrt{(\beta c)^2-1}\;\gamma^{-1}}\\
& +\frac{2\sqrt{2}c(1+\beta c)^{3/2}\,\gamma^{-1}}{\sqrt{\beta^2c^2-1}},
\end{aligned} \tag{b1}
$$

$$
\begin{aligned}
f_2(v) = {} & -\frac{2(1+\beta c)\,(\sqrt{2}\,c\sqrt{1+\beta c}+2v)}{2(\beta c-1)}+\frac{4}{\beta c-1}\,(\sqrt{2}c\sqrt{1+\beta c}+v+\beta cv)\\
& -\frac{2\sqrt{2}\sqrt{1+\beta c}\,v(-\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)}{c\sqrt{(\beta c)^2-1}\,\gamma^{-1}}\\
& -\frac{2\sqrt{2}c(1+\beta c)^{3/2}\,\gamma^{-1}}{\sqrt{\beta^2c^2-1}},
\end{aligned} \tag{b2}
$$

$$
\begin{aligned}
f_3(v) = {} & -\frac{2\beta\gamma v}{c}\sqrt{\beta^2c^2-1}\,(1-\beta v)+\beta[2(-\beta c+\beta^2c^2)\\
& +2\beta c\sqrt{(\beta c)^2-1}\,\gamma^1],
\end{aligned} \tag{b3}
$$

$$
\begin{aligned}
g_1(v) = {} & 2\sqrt{2}\,c(\beta c-1)\sqrt{1+\beta c}\left[-\frac{2(\sqrt{2}\,c\sqrt{1+\beta c}-2v)}{(\beta c-1)(-\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta vc)}\right.\\
& \left.-\frac{2\sqrt{2}\,c\sqrt{1+\beta c}\,\gamma^{-1}}{\sqrt{(\beta c)^2-1}(-\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)}\right],
\end{aligned} \tag{b4}
$$

$$
\begin{aligned}
g_2(v) = {} & 2\sqrt{2}\,c(\beta c-1)\sqrt{1+\beta c}\left[\frac{2(\sqrt{2}\,c\sqrt{1+\beta c}-2v)}{(\beta c-1)(\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta vc)}\right.\\
& \left.+\frac{2\sqrt{2}\,c\sqrt{1+\beta c}\,\gamma^{-1}}{\sqrt{(\beta c)^2-1}(\sqrt{2}\,c\sqrt{1+\beta c}+v+\beta cv)}\right],
\end{aligned} \tag{b5}
$$

and

$$g_3(v) = v[2(-\beta c + \beta^2 c^2) + 2\beta c\sqrt{(\beta c)^2 - 1}\,\gamma^{-1}]. \tag{b6}$$

The function $A_\beta^{(2)}(v)$ is given by

$$A_\beta^{(2)}(v) = \frac{mc^2\beta}{(1-\beta^2c^2)^{3/2}}\left[\beta c\ \ln\left(\frac{\frac{2(\beta c - v/c)}{1-\beta^2c^2} + \frac{2}{\gamma\sqrt{1-\beta^2c^2}}}{1-\beta v}\right)\right.$$
$$\left. - \sqrt{1-\beta^2c^2}\ \ln\left(\frac{2c(1-\beta^2c^2 + (1-\beta^2c^2)\,\gamma^{-1}}{v(1-\beta^2c^2)^{3/2}}\right)\right]. \tag{B2}$$

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